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Toward an understanding of entanglement for generalized n-qubit W-states

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Abstract
We solve stationarity equations of the geometric measure of entanglement for multi-qubit W-type states. In this way we compute analytically the maximal overlap of one-parameter n-qubit and two-parameter four-qubit W-type states and their nearest product states. Possible extensions to arbitrary W-type states and geometrical interpretations of these results are discussed in detail.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement of quantum states [1] plays an important role in quantum information, computation and communication (QICC). It is a genuine physical resource for the teleportation process [2, 3] and makes it possible that the quantum computer outperforms the classical one [4, 5]. It also plays a crucial role in quantum cryptographic schemes [6, 7]. These phenomena have provided the basis for the development of modern quantum information science.

Quantum entanglement is a rich field of research. A better understanding of quantum entanglement, of ways it is characterized, created, detected, stored and manipulated, is theoretically the most basic task of the current QICC research. In the bipartite case entanglement is relatively well understood, while in the multipartite case even quantifying the entanglement of pure states is a great challenge.

The geometric measure of entanglement can be considered as one of the most reliable quantifiers of multipartite entanglement [8–10]. It depends on $P_{\max}$, the maximal overlap of a given state with the nearest product state, and is defined by the formula $E_g(\psi) = 1 - P_{\max}$ [10]. The same overlap $P_{\max}$, known also as the injective tensor norm of ψ [11], is the maximal probability of success in Grover's search algorithm [12] when the state ψ is used as an input state. This relationship between the success probability of the quantum search algorithm and the amount of entanglement of the input state allows one to define an operational entanglement measure known as Groverian entanglement [13, 14].

The maximal overlap $P_{\max}$ is a useful quantity and has several practical applications. It has been used to study quantum phase transitions in spin models [15, 16] and to quantify the distinguishability of multipartite states by local means [17]. Moreover, $P_{\max}$ exhibits interesting connections with entanglement witnesses and can be efficiently estimated in experiments [18]. Recently, it has been shown that the maximal overlap is the largest coefficient of the generalized Schmidt decomposition and the nearest product state uniquely defines the factorizable basis of the decomposition [19, 20].

In spite of its usefulness one obstacle to use $P_{\max}$ fully in quantum information theories is the fact that it is difficult to compute it analytically for generic states. The usual maximization method generates a system of nonlinear equations [10]. Thus, it is important to develop a technique for the computation of $P_{\max}$ [21–25].

Theorem I of [21] enables us to compute $P_{\max}$ for n-qubit pure states by making use of $(n-1)$-qubit reduced states. In the case of three-qubit states the theorem effectively changes the nonlinear eigenvalue equations into the linear form. Owing to this essential simplification $P_{\max}$ for the generalized three-qubit W-state [26, 27] was computed analytically in [28]. Furthermore, in [29] $P_{\max}$ was found for three-qubit quadrilateral states with an elegant geometric interpretation. More recently, based on the analytical results of [28, 29] and the classification of [30], $P_{\max}$ for various types of three-qubit states was computed analytically and expressed in terms of local unitary (LU) invariants [31].

In general, the calculation of the multi-partite entanglement is confronted with great difficulties. Furthermore, even if we know the explicit expressions of entanglement measure, the separation of the applicable domains is also a nontrivial task [29]. Therefore, there is a good reason to consider first some solvable cases that allow analytic solutions and clear separations of the validity domains. Later, these results could be extended, either analytically or numerically, for a wider class of multi-qubit states. In light of these ideas we consider one- and two-parametric n-qubit W-type states with $n \geqslant 4$ in this paper.

The paper is organized as follows. In section 2, we clarify our tasks and notations. In section 3, we review the calculational tool introduced in [21, 28, 29] and explain how the Lagrange multiplier method gives simple solution to the one-parameter cases. This method is used in section 4 for the derivation of $P_{\max}$ for one-parameter W-states in four-qubit, five-qubit and six-qubit systems. In this section the analytical results are compared with numerical data. In section 5, based on the analytical results of the previous section we compute $P_{\max}$ for a one-parameter W-state in an arbitrary n-qubit system. In section 6, we derive $P_{\max}$ for two-parameter W-states in the four-qubit system by adopting the usual maximization technique. In section 7, we analyze two-parameter results by considering several particular cases. In section 8, we discuss the possibility of extensions of the results to arbitrary W states and the existence of a geometrical interpretation.

2. Summary of tasks

Let $|\psi\rangle$ be a pure state of an n-party system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where the dimensions of the individual state spaces $\mathcal{H}_k$ are finite but otherwise arbitrary. The maximal overlap of $|\psi\rangle$ is given by

$$P_{\max}(\psi) \equiv \max_{|q_1\rangle\cdots|q_n\rangle} |\langle q_1|\langle q_2|\cdots\langle q_n|\psi\rangle|^2, \tag{1}$$

where the maximum is taken over all single-system normalized state vectors $|q_k\rangle \in \mathcal{H}_k$, and it is understood that $|\psi\rangle$ is normalized.

Let us consider now an n-qubit W-type state

$$|W_n\rangle = a_1|10\cdots 0\rangle + a_2|010\cdots 0\rangle + \cdots + a_n|0\cdots 01\rangle, \tag{2}$$

where the labels within each ket refer to qubits $1, 2, \ldots, n$ in that order. Without loss of generality we consider only the case of positive parameters a_i with $i = 1, 2, \ldots, n$.

In this paper we will compute analytically $P_{\max}$ in the following two cases:

(1) for the one-parametric $|W_n\rangle$ when $a_1 = \cdots = a_{n-1} \equiv a$ and $a_n \equiv q$

(2) for the two-parametric $|W_4\rangle$ when $a_1 = a,\ a_2 = b,\ a_3 = a_4 = q$.

To ensure the calculational validity we use the result of [14], which has shown that $P_{\max} = (1 - 1/n)^{n-1}$ when $a_1 = a_2 = \cdots = a_n$. Thus, the final results of the one-parametric case should agree with the following.

• If $a = q = 1/\sqrt{n}$, then $P_{\max}$ should be equal to $(1 - 1/n)^{n-1}$.

• If $q = 0$, then $|W_n\rangle$ becomes $|W_{n-1}\rangle \otimes |0\rangle$ and, as a result, $P_{\max}$ should be equal to $(1 - 1/(n-1))^{n-2}$.

For the two-parametric case, $P_{\max}(W_4)$ should have a correct limit when either a or b vanishes. At $a = 0$ we have $|W_4\rangle = |0\rangle \otimes |W_3\rangle$, and thus the maximal overlap should be expressed in terms of the circumradius of the isosceles triangle with sides b, q, q [28].

3. Calculation tool

For a pure state of two qubits $P_{\max}$ is given by

$$P_{\max} = \frac{1}{2}[1 + \sqrt{1 - 4\det\rho^A}], \tag{3}$$

where ρ^A is the reduced density matrix, i.e. $\mathrm{Tr}_B\rho^{AB}$. Therefore, the Bell (and their LU-equivalent) states have the minimal ($P_{\max} = 1/2$) while the product states have the maximal ($P_{\max} = 1$) overlap.

The explicit dependence of $P_{\max}$ on state parameters for the generalized three-qubit W-state,

$$|W_3\rangle = a_1|100\rangle + a_2|010\rangle + a_3|001\rangle, \tag{4}$$

was computed in [28]. In order to express explicitly $P_{\max}(W_3)$ in terms of state parameters, we define a set $\{\alpha, \beta, \gamma\}$ as the set $\{a_1, a_2, a_3\}$ in decreasing order. Then $P_{\max}$ for the generalized W-state can be expressed in a form

$$P_{\max}(W_3) = \begin{cases} 4R_W^2 & \text{when } \alpha^2 \leqslant \beta^2 + \gamma^2 \\ \alpha^2 & \text{when } \alpha^2 \geqslant \beta^2 + \gamma^2, \end{cases} \tag{5}$$

where R_W is the circumradius of the triangle with sides a_1, a_2, a_3. Similar calculation procedure can be applied to the three-qubit quadrilateral state. It has been shown in [29] that for this case $P_{\max}$ is expressed in terms of the circumradius of a convex quadrangle. These two separate results strongly suggest that $P_{\max}$ for an arbitrary pure state has its own geometrical meaning. If we are able to know this meaning completely, then our understanding on the multipartite entanglement would be greatly enhanced.

Now, we briefly review how to derive the analytic result (5) because it plays crucial role in next two sections. In [28], $P_{\max}$ for three-qubit state is expressed as

$$P_{\max} = \frac{1}{4}\max_{|\vec{s}_1|=|\vec{s}_2|=1}[1 + \vec{s}_1\cdot\vec{r}_1 + \vec{s}_2\cdot\vec{r}_2 + g_{ij}s_{1i}s_{2j}], \tag{6}$$

where $\vec{s}_1$ and $\vec{s}_2$ are Bloch vectors of the single-qubit states. In equation (6) $\vec{r}_1 = \mathrm{Tr}[\rho^A \vec{\sigma}]$, $\vec{r}_2 = \mathrm{Tr}[\rho^B \vec{\sigma}]$ and $g_{ij} = \mathrm{Tr}[\rho^{AB} \sigma_i \otimes \sigma_j]$, where ρ^A, ρ^B and ρ^{AB} are appropriate partial traces of $\rho^{ABC} \equiv |W_3\rangle\langle W_3|$ and σ_i are usual Pauli matrices. The explicit expressions of $\vec{r}_1$, $\vec{r}_2$ and g_{ij} are given in [28]. Due to maximization over $\vec{s}_1$ and $\vec{s}_2$ in equation (6), we can compute $\vec{s}_1$ and $\vec{s}_2$ by solving the Lagrange multiplier equations

$$\vec{r}_1 + g\vec{s}_2 = \lambda_1 \vec{s}_1, \qquad \vec{r}_2 + g^T \vec{s}_1 = \lambda_2 \vec{s}_2, \tag{7}$$

where λ_1 and λ_2 are Lagrange multiplier constants. Now, we let $s_{1y} = s_{2y} = 0$ for simplicity because they give only irrelevant overall phase factor to $\langle q_1|\langle q_2|\langle q_3|W_3\rangle$. After eliminating the Lagrange multiplier constants, one can show that equation (7) reduces to two equations. Examining these two remaining equations, one can show that $\vec{s}_1$ and $\vec{s}_2$ have the following relation to each other:

$$\vec{s}_1(a_1, a_2, a_3) = \vec{s}_2(a_2, a_1, a_3). \tag{8}$$

Using this relation, one can combine these two equations into single one expressed in terms of solely s_{1z} in a final form

$$\frac{\sqrt{1 - s_{1z}^2(a_1, a_2, a_3)}}{s_{1z}(a_1, a_2, a_3)} = \frac{\omega\sqrt{1 - s_{1z}^2(a_2, a_1, a_3)}}{r_1 - r_3 s_{1z}(a_2, a_1, a_3)}, \tag{9}$$

where $r_1 = a_2^2 + a_3^2 - a_1^2$, $r_2 = a_1^2 + a_3^2 - a_2^2$, $r_3 = a_1^2 + a_2^2 - a_3^2$ and $\omega = 2a_1 a_2$. Defining $a_1 = a_2 \equiv a$ and $a_3 \equiv q$ again, one can solve equation (9) easily in a form

$$\begin{aligned} s_{1z} &= s_{2z} = \frac{r_1}{\omega + r_3} = \frac{q^2}{4a^2 - q^2} \\ s_{1x} &= s_{2x} = \sqrt{1 - s_{1z}^2} = \frac{2\sqrt{2}a}{4a^2 - q^2}\sqrt{2a^2 - q^2}. \end{aligned} \tag{10}$$

Inserting equation (10) into equation (6), one can compute $P_{\max}$ for $|W_3\rangle$ with $a_1 = a_2 = a$ and $a_3 = q$, whose final expression is simply

$$P_{\max} = \frac{(1 - q^2)^2}{2 - 3q^2}. \tag{11}$$

From equation (10) it follows that equation (11) is valid only when $q^2 \leqslant 2a^2$. This constraint defines the applicable domain of the above expression for the maximal overlap. The final expression (11) is consistent with equation (5) in the range of variation of equation (10) and has correct limits. When $q = 0$, equation (11) gives $P_{\max} = 1/2$ which corresponds to that of the two-qubit EPR state. When $q = 1/\sqrt{3}$, equation (11) gives $P_{\max} = 4/9$, which is also consistent with the result of [14].

4. Four-, five- and six-qubit W-type states: one-parametric cases

The method described in the previous section may enable us to compute $P_{\max}$ of four-qubit W-type states. For the case of arbitrary four-qubit systems $P_{\max}$ can be represented in a form

$$\begin{aligned} P_{\max} = \frac{1}{8} \max_{|\vec{s}_1|=|\vec{s}_2|=|\vec{s}_3|=1} \big[&1 + \vec{s}_1 \cdot \vec{r}_1 + \vec{s}_2 \cdot \vec{r}_2 + \vec{s}_3 \cdot \vec{r}_3 + s_{1i}s_{2j}g_{ij}^{(3)} + s_{1i}s_{3j}g_{ij}^{(2)} \\ &+ s_{2i}s_{3j}g_{ij}^{(1)} + s_{i1}s_{2j}s_{3k}h_{ijk} \big], \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\vec{r}_1 = \mathrm{Tr}[\rho^A \vec{\sigma}], \qquad \vec{r}_2 = \mathrm{Tr}[\rho^B \vec{\sigma}], \qquad \vec{r}_3 = \mathrm{Tr}[\rho^C \vec{\sigma}], \\ &g_{ij}^{(3)} = \mathrm{Tr}[\rho^{AB} \sigma_i \otimes \sigma_j], \qquad g_{ij}^{(2)} = \mathrm{Tr}[\rho^{AC} \sigma_i \otimes \sigma_j], \qquad g_{ij}^{(1)} = \mathrm{Tr}[\rho^{BC} \sigma_i \otimes \sigma_j] \\ &h_{ijk} = \mathrm{Tr}[\rho^{ABC} \sigma_i \otimes \sigma_j \otimes \sigma_k]. \end{aligned} \tag{13}$$

For the case of the generalized four-qubit W-state all vectors $\vec{r}_k$ are collinear, all matrices $g^{(k)}$ are diagonal and the vectors $\vec{r}_k$ are eigenvectors of the matrices $g^{(k)}$ as follows:

$$\vec{r}_k = (0, 0, r_k), \qquad g_{ij}^{(k)} = \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & \omega_k & 0 \\ 0 & 0 & -\tilde{r}_k \end{pmatrix}, \qquad k = 1, 2, 3. \tag{14}$$

In equation (14) we defined various quantities as follows:

$$\begin{aligned} & r_k = a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_k^2, \\ & \omega_1 = 2a_2a_3, \qquad \omega_2 = 2a_1a_3, \qquad \omega_3 = 2a_1a_2. \\ & \tilde{r}_1 = a_2^2 + a_3^2 - a_1^2 - a_4^2, \qquad \tilde{r}_2 = a_1^2 + a_3^2 - a_2^2 - a_4^2, \\ & \tilde{r}_3 = a_1^2 + a_2^2 - a_3^2 - a_4^2. \end{aligned} \tag{15}$$

In addition, the non-vanishing components of h_{ijk} are

$$h_{113} = h_{223} = \omega_3 \qquad h_{131} = h_{232} = \omega_2 \qquad h_{311} = h_{322} = \omega_1 \qquad h_{333} = -r_4. \tag{16}$$

Due to the maximization in equation (12) the Bloch vectors should satisfy the following Lagrange multiplier equations:

$$\begin{aligned} & r_{1i} + g_{ij}^{(3)} s_{2j} + g_{ij}^{(2)} s_{3j} + h_{ijk} s_{2j} s_{3k} = \Lambda_1 s_{1i} \\ & r_{2i} + g_{ji}^{(3)} s_{1j} + g_{ij}^{(1)} s_{3j} + h_{kij} s_{1k} s_{3j} = \Lambda_2 s_{2i} \\ & r_{3i} + g_{ji}^{(2)} s_{1j} + g_{ji}^{(1)} s_{2j} + h_{jki} s_{1j} s_{2k} = \Lambda_3 s_{3i}. \end{aligned} \tag{17}$$

Now we put $s_{1y} = s_{2y} = s_{3y} = 0$ as before. After removing the Lagrange multiplier constants Λ_1, Λ_2 and Λ_3, one can show that equation (17) reduce to the following three equations:

$$\begin{aligned} & s_{1x}\left[r_1 - \tilde{r}_3 s_{2z} - \tilde{r}_2 s_{3z} + \omega_1 s_{2x} s_{3x} - r_4 s_{2z} s_{3z}\right] = s_{1z}\left[\omega_2 s_{3x}(1 + s_{2z}) + \omega_3 s_{2x}(1 + s_{3z})\right] \\ & s_{2x}\left[r_2 - \tilde{r}_3 s_{1z} - \tilde{r}_1 s_{3z} + \omega_2 s_{1x} s_{3x} - r_4 s_{1z} s_{3z}\right] = s_{2z}\left[\omega_1 s_{3x}(1 + s_{1z}) + \omega_3 s_{1x}(1 + s_{3z})\right] \\ & s_{3x}\left[r_3 - \tilde{r}_1 s_{2z} - \tilde{r}_2 s_{1z} + \omega_3 s_{1x} s_{2x} - r_4 s_{1z} s_{2z}\right] = s_{3z}\left[\omega_2 s_{1x}(1 + s_{2z}) + \omega_1 s_{2x}(1 + s_{1z})\right]. \end{aligned} \tag{18}$$

Equation (18) implies that the Bloch vectors have the following symmetries:

$$\begin{aligned} & \vec{s}_1(a_1, a_2, a_3, a_4) = \vec{s}_2(a_2, a_1, a_3, a_4) = \vec{s}_3(a_3, a_2, a_1, a_4) \\ & \vec{s}_1(a_1, a_2, a_3, a_4) = \vec{s}_1(a_1, a_3, a_2, a_4) \\ & \vec{s}_2(a_1, a_2, a_3, a_4) = \vec{s}_2(a_3, a_2, a_1, a_4) \\ & \vec{s}_3(a_1, a_2, a_3, a_4) = \vec{s}_3(a_2, a_1, a_3, a_4). \end{aligned} \tag{19}$$

Therefore, one can compute all Bloch vectors if one of them is known. Using the symmetries (19), we can make single equation from equation (18) which is expressed in terms of s_{1z} only in a form

$$\frac{s_{1x}(a_1, a_2, a_3, a_4)}{s_{1z}(a_1, a_2, a_3, a_4)} = \frac{P(a_1, a_2, a_3, a_4)}{Q(a_1, a_2, a_3, a_4)} \tag{20}$$

where

$$\begin{aligned} P(a_1, a_2, a_3, a_4) = {} & \omega_2 \sqrt{1 - s_{1z}^2(a_3, a_2, a_1, a_4)}\,\left[1 + s_{1z}(a_2, a_1, a_3, a_4)\right] \\ & + \omega_3 \sqrt{1 - s_{1z}^2(a_2, a_1, a_3, a_4)}\,\left[1 + s_{1z}(a_3, a_2, a_1, a_4)\right] \\ Q(a_1, a_2, a_3, a_4) = {} & r_1 - \tilde{r}_3 s_{1z}(a_2, a_1, a_3, a_4) - \tilde{r}_2 s_{1z}(a_3, a_2, a_1, a_4) \\ & + \omega_1 \sqrt{1 - s_{1z}^2(a_2, a_1, a_3, a_4)} \sqrt{1 - s_{1z}^2(a_3, a_2, a_1, a_4)} \\ & - r_4 s_{1z}(a_2, a_1, a_3, a_4) s_{1z}(a_3, a_2, a_1, a_4). \end{aligned}$$

Defining $a_1 = a_2 = a_3 \equiv a$ and $a_4 \equiv q$, one can solve equation (20) easily. The final expressions of solutions are

$$\begin{aligned} s_{1z} &= s_{2z} = s_{3z} = \frac{1}{9a^2 - q^2} \\ s_{1x} &= s_{2x} = s_{3x} = \sqrt{1 - s_{1z}^2} = \frac{2\sqrt{6}a}{9a^2 - q^2}\sqrt{3a^2 - q^2}. \end{aligned} \tag{21}$$

Inserting equation (21) into equation (12), one can compute $P_{\max}$ for $|W_4\rangle$ with $a_1 = a_2 = a_3 \equiv a$ and $a_4 \equiv q$ whose final expression is

$$P_{\max} = \frac{2^2(1-q^2)^3}{(3-4q^2)^2}. \tag{22}$$

Equation (21) implies that $P_{\max}$ in equation (22) is valid when $q^2 \leqslant 3a^2$. When $q = 0$, $P_{\max}$ becomes $4/9$ as expected. When $q = 1/2$, $P_{\max}$ becomes $27/64$, which is in agreement with the result of [14].

One can repeat the calculation for $|W_5\rangle$ with $a_1 = a_2 = a_3 = a_4 \equiv a$ and $a_5 = q$. Then the final expression of $P_{\max}$ becomes

$$P_{\max} = \frac{3^3(1-q^2)^4}{(4-5q^2)^3}. \tag{23}$$

When $q = 0$, $P_{\max}$ reduces to $27/64$ as expected. When $q = 1/\sqrt{5}$, $P_{\max}$ reduces to $(4/5)^4$. By the same way $P_{\max}$ for $|W_6\rangle$ can be written as

$$P_{\max} = \frac{4^4(1-q^2)^5}{(5-6q^2)^4}. \tag{24}$$

Figure 1 is a plot of the q-dependence of $P_{\max}$ for $|W_4\rangle$, $|W_5\rangle$ and $|W_6\rangle$. The black dots are numerical data computed by the numerical technique exploited in [14]. The red solid and red dotted lines are equation (22), equation (23) and equation (24) when $q \leqslant 1/\sqrt{2}$ and $q \geqslant 1/\sqrt{2}$, respectively. As expected, the numerical data are in perfect agreement with equation (22), equation (23) and equation (24) in the applicable domain, i.e. $q^2 \leqslant (n-1)a^2$ for $|W_n\rangle$. Outside the applicable domain ($q \geqslant 1/\sqrt{2}$) the numerical data are in disagreement with these equations. The numerical analysis strongly suggests that at the outside of the applicable domains $P_{\max}$ becomes $\max(a^2, q^2)$.

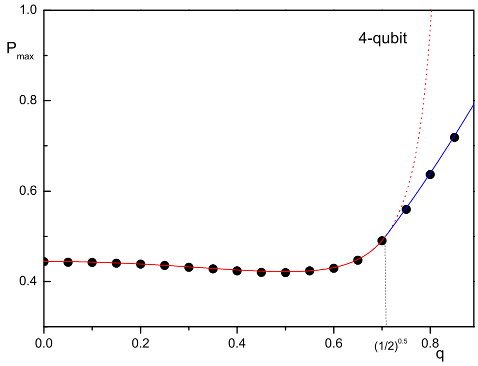
5. General multi-qubit W-type states: one-parametric cases

From equations (11), (22), (23) and (24), one can guess that $P_{\max}$ for the one-parameter W_n ($a_1 = \cdots = a_{n-1} \equiv a$, $a_n \equiv q$) is

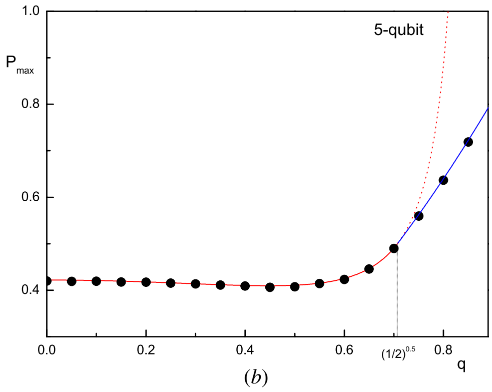
$$P_{\max}(n, q) = (1-q^2)^{n-1}\left(\frac{n-2}{(n-1) - nq^2}\right)^{n-2}. \tag{25}$$

For now we assume that $P_{\max}$ given by equation (25) is the maximal overlap. Using it, one can straightforwardly construct the nearest product state to $|W_n\rangle$. After some algebra, when $q^2 \leqslant (n-1)a^2$, one can show that the analytic expression of the nearest product state is $|q_1\rangle \otimes |q_2\rangle \otimes \cdots \otimes |q_n\rangle$, where

$$\begin{aligned} |q_1\rangle &= \cdots = |q_{n-1}\rangle \\ &= \frac{1}{\sqrt{(n-1)^2a^2 - q^2}}\left[\sqrt{(n-1)(n-2)}a|0\rangle + \sqrt{(n-1)a^2 - q^2}\, \mathrm{e}^{\mathrm{i}\varphi}|1\rangle\right] \\ |q_n\rangle &= \frac{1}{\sqrt{(n-1)^2a^2 - q^2}}\left[\sqrt{(n-1)^2a^2 - (n-1)q^2}|0\rangle + \sqrt{n-2}q\, \mathrm{e}^{\mathrm{i}\varphi}|1\rangle\right] \end{aligned} \tag{26}$$



(*a*)



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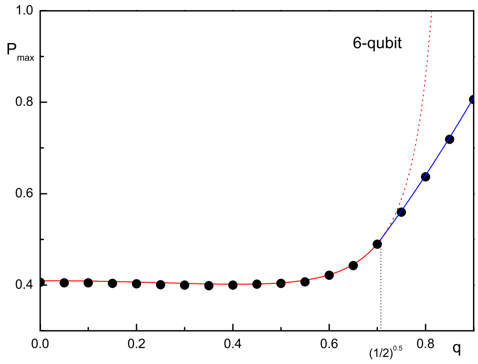

(*c*)

Figure 1. Plot of the q-dependence of $P_{\max}$ for four-qubit (*a*), five-qubit (*b*) and six-qubit (*c*). The dots are numerical data of $P_{\max}$. The solid lines are result of equation (25) in the applicable domain, $0 \leqslant q \leqslant 1/\sqrt{2}$. The dotted lines are result of equation (25) outside the applicable domain. The solid lines at $q \geqslant 1/\sqrt{2}$ are plot of $\max(a^2, q^2) = q^2$ outside the applicable domain. These figures strongly suggest that $P_{\max}$ for $|W_n\rangle$ is equation (25) when $q \leqslant 1/\sqrt{2}$ and $\max(a^2, q^2) = q^2$ when $q \geqslant 1/\sqrt{2}$.

and φ is an arbitrary phase factor. When $q^2 \geqslant (n-1)a^2$, the nearest product state, of course, becomes $|0\cdots01\rangle$.

It remains to show that expression (25) is a global maximum of the overlap. We do not present a rigorous proof, but rather present convincing arguments instead.

First, it is easy to check that

$$\langle q_1 q_2 \cdots q_{n-1} | W_n\rangle = \mathrm{e}^{-\mathrm{i}\varphi}\sqrt{P_{\max}}|q_n\rangle, \qquad \langle q_2 q_3 \cdots q_{n-1} q_n | W_n\rangle = \mathrm{e}^{-\mathrm{i}\varphi}\sqrt{P_{\max}}|q_1\rangle. \tag{27}$$

The second equation in (27) is invariant under the permutations ($q_1 \leftrightarrow q_j$, $j = 2, 3, \ldots, n-1$). Thus, the product state equation (26) satisfies the stationarity equations of [10], and consequently, $P_{\max}$ given by equation (25) is either a local or the global maximum of the overlap of the state W_n.

Second, equation (25) gives the global maximum of the overlap in the following particular cases: $n = 3$, $n = 4$, $q = 0$ and $q = a$. Any local maximum cannot reproduce the global maximum at all these particular limits. Therefore equation (25) is the injective tensor norm of $|W_n\rangle$. Accordingly, local states (equation (26)) are the constituents of the nearest separable state.

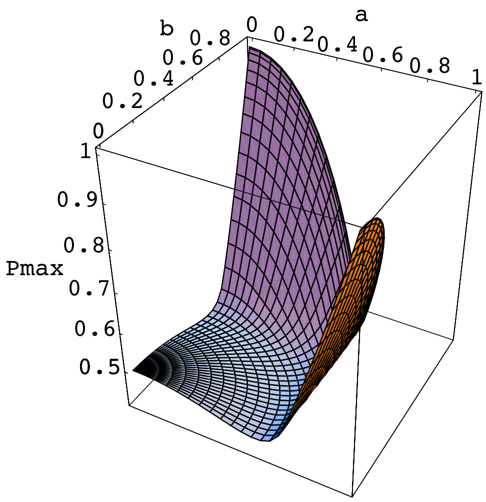

Figure 2. The maximal overlap $P_{\max}$ versus the parameters a and b for the four-qubit state. The lower part of the surface is the highly entangled regions and the maximal overlap is given by equation (39). The remaining area is a slightly entangled region and the maximal overlap is $\max(a^2, b^2)$. It is minimal ($P_{\max} = 27/64$) at $a = b = 1/2$ which is the W-state and maximal ($P_{\max} = 1$) either at $a = 1, b = 0$ or at $a = 0, b = 1$ which are the product states.

When $q = 0$ and $q = 1/\sqrt{n}$, $P_{\max}$ reduces to $(1 - 1/(n-1))^{n-2}$ and $(1 - 1/n)^{n-1}$, respectively. Thus, equation (25) is perfectly in agreement with the result of [14]. Another interesting point in equation (25) is that $P_{\max}$ becomes $1/2$ regardless of n when $q = 1/\sqrt{2}$, the boundary of the applicable domain. This makes us conjecture that outside the applicable domain $P_{\max}$ becomes $\max(a^2, q^2) = q^2$ like the three-qubit case. The blue solid lines in figure 1 are the plot of q^2 at the domain $q \geqslant 1/\sqrt{2}$. As we conjecture, the blue lines are perfectly in agreement with the numerical data. This agreement with equation (5) enables us to conjecture again that $P_{\max}$ for the n-qubit W-state given in equation (2) is $\max(a_1^2, a_2^2, \ldots, a_n^2)$ when $\alpha_1^2 \geqslant \alpha_2^2 + \cdots + \alpha_n^2$, where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is $\{a_1, a_2, \ldots, a_n\}$ with decreasing order.

Another consequence of equation (25) is the entanglement witness $\hat{W}_n$ for a one-parametric W-type state [10]. Its construction is straightforward as the following form:

$$\hat{W}_n = P_{\max}(n, q)\mathbb{1} - |W_n(q)\rangle\langle W_n(q)|, \tag{28}$$

where $\mathbb{1}$ is a unit matrix. Obviously one can show

$$\mathrm{Tr}\left(\hat{W}_n|W_n(q)\rangle\langle W_n(q)|\right) < 0, \quad \mathrm{Tr}\left(\hat{W}_n\rho_0\right) \geqslant 0, \tag{29}$$

where ρ_0 is any separable state. Thus, $\hat{W}_n$ is an entanglement witness and allows an experimental detection of the multipartite entanglement.

6. Four-qubit W state: two-parametric cases

In this section we will compute $P_{\max}$ for the two-parametric $|W_4\rangle$ given by

$$|W_4\rangle = a|1000\rangle + b|0100\rangle + q|0010\rangle + q|0001\rangle. \tag{30}$$

It seems to be difficult to apply the Lagrange multiplier method directly due to their non-trivial nonlinearity. Thus, we will adopt the usual maximization method.

The maximum overlap probability $P_{\max}$ is

$$P_{\max} = \max_{|q_1\rangle|q_2\rangle|q\rangle} |\langle q_1|\langle q_2|\langle q|\langle q \mid W_4\rangle|^2. \tag{31}$$

Now we define the one-qubit states as $|q_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, $|q_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$ and $|q\rangle = \gamma_0|0\rangle + \gamma_1|1\rangle$. For simplicity, we assume that all coefficients are real and positive. Then, $P_{\max}$ becomes

$$P_{\max} = \max_{\alpha_0,\beta_0,\gamma_0} \gamma_0^2\left(a\beta_0\gamma_0\sqrt{1-\alpha_0^2} + b\alpha_0\gamma_0\sqrt{1-\beta_0^2} + 2q\alpha_0\beta_0\sqrt{1-\gamma_0^2}\right)^2. \tag{32}$$

Since the maximum value is determined at the extremum point, it is useful if the extremum conditions are derived. This is achieved by differentiating equation (32), which leads to

$$\begin{aligned}
&b\gamma_0\sqrt{1-\beta_0^2} + 2q\beta_0\sqrt{1-\gamma_0^2} = a\beta_0\gamma_0\frac{\alpha_0}{\sqrt{1-\alpha_0^2}}\\
&a\gamma_0\sqrt{1-\alpha_0^2} + 2q\alpha_0\sqrt{1-\gamma_0^2} = b\alpha_0\gamma_0\frac{\beta_0}{\sqrt{1-\beta_0^2}}\\
&a\beta_0\gamma_0\sqrt{1-\alpha_0^2} + b\alpha_0\gamma_0\sqrt{1-\beta_0^2} + q\alpha_0\beta_0\sqrt{1-\gamma_0^2} = q\alpha_0\beta_0\frac{\gamma_0^2}{\sqrt{1-\gamma_0^2}}.
\end{aligned} \tag{33}$$

One can solve the equations by separating α_0 from β_0, γ_0, i.e.

$$\begin{aligned}
&\frac{\alpha_0}{\sqrt{1-\alpha_0^2}} = \frac{b}{a}\frac{\sqrt{1-\beta_0^2}}{\beta_0} + \frac{2q}{a}\frac{\sqrt{1-\gamma_0^2}}{\gamma_0}\\
&\frac{\sqrt{1-\alpha_0^2}}{\alpha_0} = \frac{b}{a}\frac{\beta_0}{\sqrt{1-\beta_0^2}} - \frac{2q}{a}\frac{\sqrt{1-\gamma_0^2}}{\gamma_0}\\
&\frac{\sqrt{1-\alpha_0^2}}{\alpha_0} = \frac{q}{a}\frac{\gamma_0}{\sqrt{1-\gamma_0^2}} - \frac{q}{a}\frac{\sqrt{1-\gamma_0^2}}{\gamma_0} - \frac{b}{a}\frac{\sqrt{1-\beta_0^2}}{\beta_0}.
\end{aligned} \tag{34}$$

Since the left-hand sides of the second and third equations in equation (34) are the same, equalizing the right-hand sides leads to an equation between β_0 and γ_0:

$$\beta_0\sqrt{1-\beta_0^2} = \frac{b}{q}\gamma_0\sqrt{1-\gamma_0^2}. \tag{35}$$

Since the left-hand side of the first equation in equation (34) is the inverse of the left-hand side in the second equation, we get the condition that multiplication of the right-hand sides must be 1. Using this condition and equation (35) we have the expression for β_0 in terms of γ_0, i.e.

$$\beta_0^2 = \frac{3}{2} - \frac{4q^2 - a^2 + b^2}{4q^2}\gamma_0^2. \tag{36}$$

Squaring both sides of equation (35) and inserting β_0^2 from equation (36), we obtain a second-order equation for γ_0^2 and the solution for γ_0 is

$$\gamma_0^2 = \frac{4q^2(4q^2 - a^2 - b^2) - 2q^2\sqrt{(4q^2 - a^2 - b^2)^2 + 12a^2b^2}}{(4q^2 + b^2 - a^2)^2 - 16q^2b^2}. \tag{37}$$

When deriving equation (37), we discard the other solution because it is not corresponding to the maximum value of $P_{\max}$. The solution for α_0 is obtained by separating β_0:

$$\alpha_0^2 = \frac{3}{2} - \frac{4q^2 + a^2 - b^2}{4q^2}\gamma_0^2. \tag{38}$$

Inserting these extremum solutions in $P_{\max}$ and rationalizing the denominator, one gets

$$P_{\max} = \frac{2q^4\left[(4q^2 - a^2 - b^2)\{(4q^2 - a^2 - b^2)^2 - 36a^2b^2\} + \{(4q^2 - a^2 - b^2)^2 + 12a^2b^2\}^{\frac{3}{2}}\right]}{\{(4q^2 - a^2 - b^2)^2 - 4a^2b^2\}^2}. \tag{39}$$

Of course, equation (39) is valid when $\alpha^2 \leqslant \beta^2 + \gamma^2 + \delta^2$, where $\{\alpha, \beta, \gamma, \delta\}$ is $\{a, b, q, q\}$ with decreasing order. When $\alpha^2 \geqslant \beta^2 + \gamma^2 + \delta^2$, $P_{\max}$ will be $\alpha^2 = \max(a^2, b^2)$.

The dependence of the maximal overlap on state parameters is shown in figure 2. The behavior of $P_{\max}$ in different limits is explained in the next section.

7. Special four-qubit W-type states

In this section we consider some special four-qubit states.

The first one is the $a = 0$ limit. Since $|W_4\rangle = |0\rangle \otimes (b|100\rangle + q|010\rangle + q|001\rangle)$ in this limit, one can compute $P_{\max}$ using equation (5). In this limit equation (39) gives

$$P_{\max} = \frac{4q^4}{4q^2 - b^2} \quad (b^2 \leqslant 2q^2). \tag{40}$$

One can easily show that this is perfectly in agreement with equation (5).

The second special case is the $a = q$ limit. In this limit equation (39) gives

$$P_{\max} = \frac{4(1 - b^2)^3}{(3 - 4b^2)^2} \quad (b^2 \leqslant 3q^2), \tag{41}$$

which is also consistent with equation (22).

The last special case is the $2q = a + b$ limit. Although both denominator and numerator in equation (39) vanish, their ratio has a finite limit and $P_{\max}$ takes correct values in the applicable domain. The applicable domain is defined by the two restrictions $\alpha^2 \leqslant \beta^2 + \gamma^2 + \delta^2$ and $2q = a + b$. These restrictions together with the normalization condition impose upper and lower bounds for the parameters a and b:

$$\min(a, b) \geqslant \frac{\sqrt{2}}{6}, \quad \max(a, b) \leqslant \frac{\sqrt{2}}{2}. \tag{42}$$

The maximum overlap probability $P_{\max}$ is

$$P_{\max} = \frac{27}{256}\frac{(a + b)^4}{ab}. \tag{43}$$

The limit $a = b = q = 1/2$ again yields $P_{\max} = 27/64$. Another interesting limit is the case when $b(a)$ is minimal and $a(b)$ is maximal. This limit is reached at $a = 3b(b = 3a)$. Then equation (43) yields $P_{\max} = 1/2 = \alpha^2$. These states are the first-type shared states [29] and allow perfect teleportation and superdense coding scenario.

8. Discussion

We have calculated the maximal overlap of one- and two-parametric W-type states and found their nearest separable states. However, in some sub-regions of the parameter space one can find the nearest states and corresponding maximal overlaps for generic W-type states. In fact, the square of any coefficient in equation (2) is a maximal overlap in some regions of state parameters. It is easy to check that the product state $|0_1 \ldots 0_{k-1} 1_k 0_{k+1} \ldots 0_n\rangle$ is a solution of the stationarity equation with an entanglement eigenvalue $\sqrt{P_{\max}} = a_k$. From the previous results one can guess that this solution gives a true maximum of the overlap if

$$a_k^2 \geqslant a_1^2 + a_2^2 + \cdots + a_{k-1}^2 + a_{k+1}^2 + \cdots + a_n^2 = 1 - a_k^2. \tag{44}$$

Then the maximal overlap in the slightly entangled region can be written readily in the form

$$P_{\max} = \max\left(a_1^2, a_2^2, \ldots, a_n^2\right) \qquad \text{if} \quad \max\left(a_1^2, a_2^2, \ldots, a_n^2\right) \geqslant \tfrac{1}{2}. \tag{45}$$

This formula has the following simple interpretation. Equation (44) means that the state is already written in the generalized Schmidt normal form and the maximal overlap takes the value of the largest coefficient [20].

Now the question at issue is what happens if $a_k^2 < 1/2,\ k = 1, 2, \ldots, n$. From these inequalities it follows that

$$\tfrac{1}{2}\,(a_1 + a_2 + \cdots + a_n) > \max(a_1, a_2, \ldots, a_n). \tag{46}$$

From any set of such coefficients one can form polygons (polyhedrons). This fact is indirect evidence that $P_{\max}$ has a geometrical meaning. Unfortunately, there is an obstacle to achieving this goal. The problem is that we do not have any answer for generic states. For example, it is difficult to conclude from equation (11) that the expression is the circumradius of a triangle in a particular limit. In general, one can form many polygons, either convex or crossed, from the set $a_1, a_2, \ldots, a_n$. Each of them generates its own geometric quantities that can be treated as the maximal overlap. This happens because stationarity equations have many solutions in the highly entangled region. And all of these solutions yield the same expression in particular cases. For example, in [29] it was shown that all convex and crossed quadrangles are contracted to the same triangle in particular limits. In conclusion, in order to find a true geometric interpretation one has to derive $P_{\max}$ for generic states.

Another (and probably promising) way to get the desired interpretation is the following. Since the surface $\left(a_1^2 - 1/2\right)\left(a_2^2 - 1/2\right) \cdots \left(a_n^2 - 1/2\right) = 0$ separates highly and slightly entangled regions, one may ask what happens on this surface. That is, we consider polygons whose sides satisfy the equality $a_k^2 = a_1^2 + a_2^2 + \cdots + a_{k-1}^2 + a_{k+1}^2 + \cdots + a_n^2$ for any k. For $n = 3$ we perfectly know that the corresponding polygons are right triangles and the center of a circumcircle lies on the largest side of a right triangle. Then, we can conclude that if the center of the circumcircle is inside the triangle, then the maximal overlap is the circumradius and otherwise is the largest coefficient. However, for $n \geqslant 4$ we do not know what are the polygons for which the square of the largest side is the sum of squares of the remaining coefficients. If one understands the geometric meaning of this relation, then one finds a clue. And this clue may enable us to find $P_{\max}$ for generic W-type states. These types of analytic expressions can have practical application in QICC and may shed new light on multipartite entanglement.

All above-mentioned problems owe their origin to the fact that the injective tensor norm is related to Cayley's hyperdeterminant [23]. It is well known that this hyperdeterminant has a geometrical interpretation for $n = 3$ and no such interpretation is known for $n \geqslant 4$ so far. We hope to keep on studying this issue in the future.

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